Analysis IV

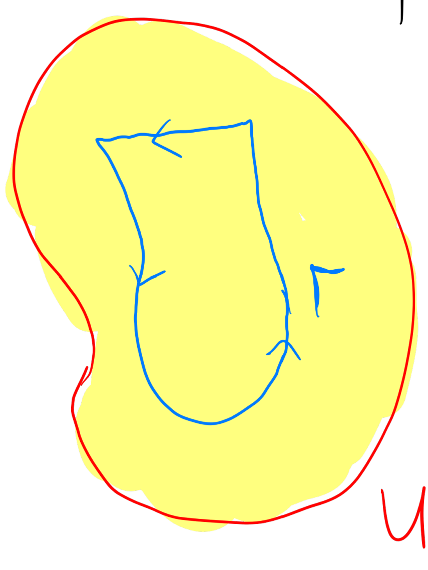
(for EL, GM, MX)

Week 4 - 10.03.25

Recall from last time:

Cauchy's theorem: If $U \subseteq \mathbb{C}$ is open and simply connected, $f: U \longrightarrow \mathbb{C}$ is holomorphic and Γ is a piecewise regular, closed curve inside U, then

$$\int_\Gamma f(z)\,dz = 0.$$



- In practice: Given f, Γ, if f is holomorphic in a set slightly larger than the interior of Γ_i: I can apply the theorem.

The above theorem allows us to shift the path of integration when convenient. For instance:

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If Γ_0, Γ_1 are simple, closed, same orientation, Γ_1 is in the interior of Γ_0 and f is holomorphic in $\operatorname{int}\Gamma_0 / \operatorname{int}\Gamma_1$, then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

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If Γ_1, Γ_2 have the same endpoints, are oriented in the same direction and f is holomorphic in the region contained between Γ_1 and Γ_2, then $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$.

(by applying Cauchy's theorem for the closed loop formed by Γ_1 and $-\Gamma_2$)

Cauchy's integral formula

Let $D \subseteq \mathbb{C}$ be open, $f: D \to \mathbb{C}$ be holomorphic and $z_0 \in D$. Let γ be any simple, closed curve with $\text{int}(\gamma) \subseteq D$ such that

$z_0 \in \operatorname{int}(\gamma)$; (and γ is oriented counter-clockwise).

Then $$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\, dz.$$

Cauchy's integral formula extended:

With the same assumptions as above,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

n-th derivative

(for $n=0$: We recover the previous formula).

Idea of proof Start from

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, dz$$

and differentiate with respect to z_0

Note: On the right hand side, z_0 appears only in $\frac{1}{z - z_0}$

And use: $$\frac{d}{dz_0}\left((z - z_0)^{-k} \right) = +k \cdot (z - z_0)^{-k-1}.$$ $\square$

Remark: The extended Cauchy's formula is used to derive the statement we made earlier in class: That holomorphic functions are infinitely many times differentiable.

Example: Let $f(z) = \frac{\cos z}{z}$. Let γ be a simple closed curve, oriented counter-clockwise. What is $\int_\gamma f(z)\,dz$?

We have to consider cases; note that f is holomorphic on $\mathbb{C} \setminus \{0\}$.

- If $0 \in \gamma$, the integral is ill-defined
- If 0 is <u>not</u> in the interior of γ, then by Cauchy's theorem:

$$\int_\gamma f(z)\,dz = 0$$

- If 0 is <u>inside</u> the interior of γ:

Note that $\int_\gamma f(z)\,dz = \int_\gamma \frac{g(z)}{z}\,dz$,

where $g(z) = \cos z$ is holomorphic on $\mathbb{C}$.

By Cauchy's integral formula:

$$g(0) = \frac{1}{2\pi i} \int_\gamma \frac{g(z)}{z}\,dz$$

So $\int_\gamma f(z)\,dz = 2\pi i\, g(0) = 2\pi i$

Example 2: Let γ be the circle $\{z : |z-2| = 1\}$ oriented counter-clockwise.

Compute $\int_\gamma \frac{e^z}{(z-2)^3}\,dz$

By Cauchy's integral formula:

If $f(z) = e^z$ (holomorphic on $\mathbb{C}$)

$$\int_\gamma \frac{e^z}{(z-2)^3}\, dz = \int_\gamma \frac{f(z)}{(z-2)^{2+1}} = \left(\frac{2\pi i}{2!}\right) f^{(2)}(0)$$

$$= \pi i$$

Laurent Series (Chapter 11 from the book)

Taylor series

Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently regular function. For any $x_0 \in \mathbb{R}$, the Taylor polynomial of degree N is defined by

$$T_{x_0}^N f(x) = f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N$$

$$= \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n .$$

<u>Theorem (Taylor)</u>: If $f \in C^{N+1}(\mathbb{R})$, then $\forall x_0, x \in \mathbb{R}$: $\exists\, \theta \in (0,1)$ such that

$$f(x) = T^N_{x_0} f(x) + \frac{f^{(N+1)}(\theta x_0 + (1-\theta)x)}{(N+1)!} (x - x_0)^{N+1}$$

- In practice, θ is not known, but we can usually estimate the $(N+1)$-th derivative
- We can approximate functions by Taylor polynomials.

If f has derivatives of all orders at x_0, then we may consider the Taylor series:

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n.$$

Question: Does the Taylor series converge?
If yes, to where?

Example: Let $f(x) = e^x$, $x_0 = 0$.

Taylor series around x_0:

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

It converges to e^x (in view of the definition of e^x).

Example 2: Let $f(x) = \frac{1}{1-x}$ and $x_0 = 0$.

Taylor series at $x_0 = 0$:

Since $f^{(n)}(x) = n!\,(1-x)^{-n-1} \Rightarrow f^{(n)}(0) = n!$

So $$\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{+\infty} x^n$$

The Taylor series converges to $f(x)$ for $|x|<1$, since

(recall that) $(1+x+\dots+x^N)(1-x) = 1-x^{N+1}$

So $$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} \xrightarrow[\text{if } |x|<1]{N\to+\infty} \frac{1}{1-x}.$$

Remark: Functions f s.t. the Taylor series converges to f in a neighborhood of x_0 are called analytic.

Not all C^∞ functions are analytic,

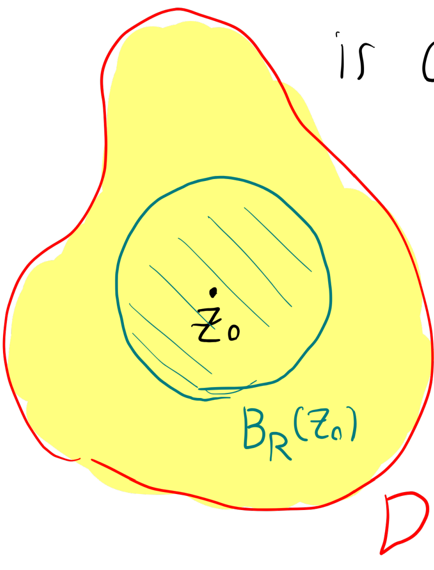
e.g. $f(x) = \begin{cases} e^{-1/x}, & x>0 \\ 0, & x\le 0 \end{cases}$ is C^∞ but not analytic.

Taylor series for complex functions / regular Laurent series.

Theorem: Let $D \subseteq \mathbb{C}$ be an open set and $z_0 \in D$. Let $R > 0$ be such that the disc

$$B_R(z_0) := \{ z : |z - z_0| < R \}$$

is contained in D.



If $f : D \to \mathbb{C}$ is <u>holomorphic</u>, then

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

for all $z \in B_R(z_0)$.

The largest such $R > 0$: <u>Radius of convergence.</u>

Remarks: 1) Holomorphic functions f are, therefore, analytic (around every z_0 in their domain, there exists a disk on which the Taylor series converges to f).

2) In practice: R is the distance of z_0 from the "nearest" point where f is not holomorphic (or not defined).

3) We can connect the Taylor series to the Cauchy integral formula:

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n=0}^{+\infty} c_n (z-z_0)^n,$$

where

$$C_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\,d\zeta$$

for any simple, closed curve γ around z_0 with $\text{int}(\gamma) \subseteq D$ (and counter-clockwise oriented).

Example 1: $f(z) = e^z$.

f is holomorphic on $\mathbb{C}$.

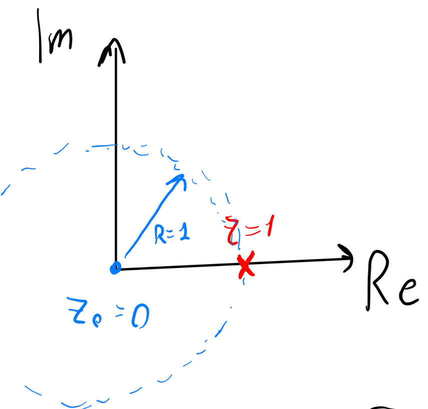
So for any z_0, the theorem applies for any $R > 0$ (so infinite radius of convergence) and we have (since $(e^z)' = e^z$):

$$e^z = \sum_{n=0}^{+\infty} e^{z_0} \frac{(z-z_0)^n}{n!}.$$

Example 2: Let $f(z) = \frac{1}{1-z}$

So $D = \mathbb{C} \setminus \{1\}$. Let $z_0 = 0$.

In this case, largest possible R: $R = 1$



Then for $z \in B_1(0)$:

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} z^n$$

Since $f^{(n)}(0) = n! \implies f(z) = \sum_{n=0}^{+\infty} z^n.$

Remark: From now on, we will be using that $\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n$ for $|z| < 1$ without reproving it each time.

Example 3: Let $f(z) = \frac{1}{1+z^2}$

Then $D = \mathbb{C} \setminus \{i, -i\}$

For $z_0 = 0$: Largest R we can choose is $R = 1$.

In $B_1(0)$: We develop Taylor series.

Using the fact that $\frac{1}{1-w} = \sum_{n=0}^{+\infty} w^n$ for $|w|<1$,

for $w = -z^2$ I get:

$$\frac{1}{1+z^2} = \sum_{n=0}^{+\infty} (-1)^n z^{2n}.$$

Note: (unique representation of Taylor series)

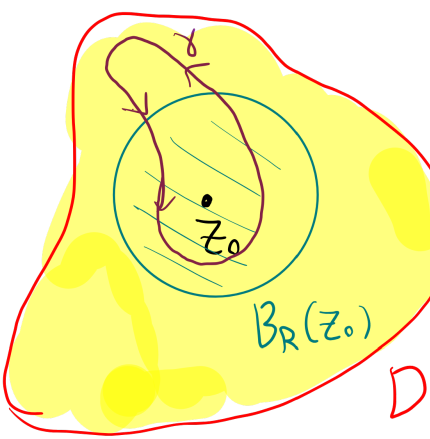
If I can express f as a series $f(z) = \sum_{n=0}^{+\infty} c_n (z - z_0)^n$, then this series has to coincide with the Taylor series (i.e. $c_n = \frac{f^{(n)}(z_0)}{n!}$). In other words, in order to compute the coefficients of the Taylor series, I'm allowed to use other tricks (e.g. use known series such as $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$ before) instead of computing all the derivatives of f at z_0.

Laurent series:

Theorem: Let $D \subseteq \mathbb{C}$ be open and $z_0 \in D$. Suppose that

$$f : D \setminus \{z_0\} \longrightarrow \mathbb{C}$$

is holomorphic. Let $R > 0$ be such that $B_R(z_0) \subseteq D$.



Then for all $z \in B_R(z_0) / \{z_0\}$:

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n \cdot (z - z_0)^n,$$

where, for any simple, closed, counter-clockwise oriented curve γ such that the interior of γ is contained in D and z_0 is in the interior of γ, the coefficients c_n are given by

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta .$$

Remarks.

- The Laurent series is like a Taylor series, but we now also include terms with "polynomial" singularities, e.g. $\frac{1}{z-z_0}$, $\frac{1}{(z-z_0)^2}$ etc.

- As in the case of the Taylor series,

the Laurent series is the <u>only</u> representation of f as a power series.

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n.$$

i.e. any other such series will have the same coefficients.

$\Longrightarrow$ If we find the coefficients by any means, not necessarily through the path integral, then we have determined the Laurent series.

<u>Definition:</u> Any Laurent series splits into a <u>singular</u> and a <u>regular</u> part.

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n \cdot (z-z_0)^n$$

$$= \underbrace{\cdots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0}}_{\text{Singular part}} + \underbrace{c_0 + c_1(z-z_0) + \cdots}_{\text{Regular part}}$$

Remark. The Taylor series is a Laurent series with no singular part.

Examples:

1) If $f: D \to \mathbb{C}$ is holomorphic and $z_0 \in D$, then

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-z_0)^n = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

is the Laurent series, which is also a Taylor series.

Note that the singular part has to vanish, since

for all $n \leq -1$: $\dfrac{f(z)}{(z-z_0)^{n+1}}$ is <u>holomorphic</u>

near z_0, so $\displaystyle\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 0$ by

Cauchy's formula.

For instance, $f(z) = e^z = \displaystyle\sum_{n=0}^{+\infty} \frac{z^n}{n!}$

is the Laurent / Taylor series of e^z around $z_0 = 0$.

2) For the following functions $f: \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$, let us find their Laurent series around $z_0 = 0$.

a) $f(z) = \dfrac{1}{z}$

Already a Laurent series:

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n, \quad c_n = \begin{cases} 1, & n = -1 \\ 0, & \text{otherwise} \end{cases}$$

b) $f(z) = \dfrac{2}{z^2}$

Similarly, $f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$,

$$c_n = \begin{cases} 2, & n = -2 \\ 0, & \text{otherwise} \end{cases}$$

c) $f(z) = \frac{5z+1}{z^2} = \frac{1}{z^2} + \frac{5}{z}$

$$= \sum_{n=-\infty}^{+\infty} c_n z^n,$$

$$c_n = \begin{cases} 1, & n = -2, \\ 5, & n = -1, \\ 0, & \text{otherwise} \end{cases}$$